

Compressed Sensing Matrices from Fourier Matrices

Guangwu Xu ^{*} and Zhiqiang Xu [†]

Abstract

The class of Fourier matrices is of special importance in compressed sensing (CS). This paper concerns deterministic construction of compressed sensing matrices from Fourier matrices. By using Katz' character sum estimation, we are able to design a deterministic procedure to select rows from a Fourier matrix to form a good compressed sensing matrix for sparse recovery. The sparsity bound in our construction is similar to that of binary CS matrices constructed by DeVore which greatly improves previous results for CS matrices from Fourier matrices. Our approach also provides more flexibilities in terms of the dimension of CS matrices. As a consequence, our construction yields an approximately mutually unbiased bases from Fourier matrices which is of particular interest to quantum information theory. This paper also contains a useful improvement to Katz' character sum estimation for quadratic extensions, with an elementary and transparent proof. Some numerical examples are included.

Keywords: ℓ_1 minimization, sparse recovery, mutual incoherence, compressed sensing matrices, deterministic construction, approximately mutually unbiased bases.

1 Introduction

In many practical situations the data of concern is sparse under suitable representations. Many problems turn to be computationally amenable under the sparsity assumption. As a notable example, it is now well understood that the ℓ_1 minimization method provides an effective way for reconstructing sparse signals in a variety of settings.

The goal of compressed sensing (CS) is to recover a sparse vector $\beta \in \mathbb{C}^N$ from the linear sampling $y = \Phi\beta$ and the sampling matrix (or compressed sensing matrix) $\Phi \in \mathbb{C}^{m \times N}$. A

^{*}Department of EE & CS, University of Wisconsin-Milwaukee, Milwaukee, WI 53211, USA; e-mail: gxu4uwm@uwm.edu. Research supported in part by the National 973 Project of China (No. 2013CB834205).

[†]Inst. Comp. Math., Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing, China; e-mail: xuzq@lsec.cc.ac.cn. Zhiqiang Xu was supported by NSFC grant 11171336 and by the Funds for Creative Research Groups of China (Grant No. 11021101).

general model can be of the form $y = \Phi\beta + z$ with z being a vector of errors (noise). In this case, one needs to approximate the sparse vector β from the linear sampling y , the compressed sensing matrix Φ and some information of z . However we shall only be interested in the noiseless case (i.e., $z = 0$) in this paper to simplify discussion.

In order to achieve less samples, one requires that the size of y be much smaller than the dimension of β , namely $m \ll N$. A naïve approach for solving this problem is to consider ℓ_0 minimization, i.e.,

$$(P_0) \quad \hat{\beta} = \arg \min_{\gamma \in \mathbb{C}^N} \{\|\gamma\|_0 \text{ subject to } y = \Phi\gamma\}. \quad (1)$$

However this is computationally infeasible. It is then natural to consider the method of ℓ_1 minimization which can be viewed as a convex relaxation of ℓ_0 minimization. The ℓ_1 minimization method in this context is

$$(P_1) \quad \hat{\beta} = \arg \min_{\gamma \in \mathbb{C}^N} \{\|\gamma\|_1 \text{ subject to } y = \Phi\gamma\}. \quad (2)$$

This method has been successfully used as an effective way for reconstructing a sparse signal in many settings.

Here, a central problem is to construct the compressed sensing matrices so that the solution to (P_1) is the same as that to (P_0) . One of the most commonly used frameworks for the compressed sensing matrices Φ is the restricted isometry property (RIP) which was introduced by Candès and Tao [8]. RIP has been used in the randomized construction of CS matrices (see for example [1, 8, 10]). Another well-known framework in compressed sensing is the mutual incoherence property (MIP) of Donoho and Huo [14]. Several deterministic constructions of CS matrices are based on MIP, e.g., [12, 24, 37].

The main focus of this paper is on the deterministic construction of CS matrices. In [12], DeVore presented a deterministic construction of CS matrices using the mutual incoherence: given a prime number p and an integer $1 < n \leq p$, an $m \times N$ (with $m = p^2, N = p^n$) binary matrix (i.e., each of its entry is either 0 or 1) Φ can be found in a deterministic manner such that whenever

$$k < \frac{\sqrt{m}}{2(n-1)} + \frac{1}{2}, \quad (3)$$

a signal β with the sparsity k (i.e., β has at most k nonzero components, we also call such a vector k -sparse) can be produced by solving (P_1) ¹. Recently, Li, Gao, Ge, and Zhang [24] suggested a deterministic construction of binary CS matrices via algebraic curves over finite fields. As stated in [24], their construction is more flexible and slightly improves DeVore's result when N is large (in fact the examples in [24] indicate that one sees improvement only when $N \geq \Theta(m^{\frac{4}{\sqrt{m}}})$).

¹In [12], it was stated that for RIP bound $\delta < 1$, it suffices that $k < \frac{\sqrt{m}}{n-1} + 1$. But for being able to recover k -sparse signal, one needs to use (3) based on the discussion in [4].

It is remarked that, in terms of construction via MIP, the bound (3) is asymptotically optimal because of the Welch lower bound [34] (details will be given in section 2). This means that one needs to require $k = O(\sqrt{m})$. Using new estimates for sumsets in product sets and for exponential sums with the products of sets possessing special additive structure, Bourgain, Dilworth, Ford, Konyagin and Kutzarva [2] were able to overcome the natural barrier $k = O(\sqrt{m})$. However there is a restriction in the construction of [2], namely the ratio $\frac{N}{m}$ is small.

Compared with binary matrices, the class of Fourier matrices is of great theoretical and practical relevance to compressed sensing. A notable technique of designing random partial Fourier matrices was suggested by Candès and Tao [9], and was improved by Rudelson and Vershynin [29]. Let $\mathcal{F}^{(N)}$ be the $N \times N$ Fourier matrix whose (k, j) -th entry is given by

$$(\mathcal{F}^{(N)})_{k,j} = \exp\left(\frac{2\pi i k j}{N}\right).$$

For $\Gamma = \frac{1}{\sqrt{N}}\mathcal{F}^{(N)}$, it was proved in [29] that if

$$m = O(k \log^4 N), \quad (4)$$

then a submatrix Φ consists of m random rows of Γ satisfying RIP conditions (which ensures k -sparse signal recovery) with high probability. This means that given N and m , the optimal k is $k = O\left(\frac{m}{\log^4 N}\right)$.

Deterministic construction of CS matrices from Fourier matrices has also received recent attention. In [35], by using the difference set, Xia, Zhou and Giannakis constructed a deterministic partial Fourier matrices with size $m \times N$ whose mutual incoherence constant meets the Welch lower bound (see Section 2). Such matrices are CS matrices for recovering a k -sparse signal whenever $k < \frac{1}{2}\left(\sqrt{\frac{(N-1)m}{N-m}} + 1\right)$. However, the restriction on the matrix size (m and N) is heavy. In fact, one can only construct such partial Fourier matrices for some very special parameters (m, N) with small $\frac{N}{m}$ (see [35]). Recently, Haupt, Applebaum, and Nowak proposed a deterministic procedure to produce a partial Fourier matrix with size $m \times N$ for a large class of (m, N) pairs [19]. More specifically, the matrices in [19] are of size $m \times N$ with N being a prime and $m \in [N^{\frac{1}{d-1}}, N]$ for some integer $d \geq 2$. This class of CS matrices can be used to recover k -sparse signal with

$$k = O\left(m^{\frac{1}{9d^2 \log d}}\right). \quad (5)$$

It is noted that there is a significant gap between the bound (5) and the (asymptotically optimal) bound (3). The allowed range of sparsity for the case of a deterministic partial Fourier matrices is far smaller than that of the binary case. Constructing partial Fourier matrices deterministically that work for a larger sparsity range of signals is certainly of particular interest.

The aim of this paper is to construct a partial Fourier matrix which is a CS matrix, through a deterministic procedure. Based on a celebrated character sum estimation of Katz [20], we are able to obtain a bound for the sparsity k in the case of partial Fourier matrices that is similar to (3). More precisely, we have shown that if $q = p^a$ is a prime power and $n > 1$, setting $N = q^n - 1$ or $N = \frac{q^n - 1}{p^b - 1} = \frac{p^{an} - 1}{p^b - 1}$ ($b \mid a$ is required for this case), then there is a deterministic process to select $m = q$ rows from the Fourier matrix $\mathcal{F}^{(N)}$ and build a (column normalized) matrix Φ , such that if

$$k < \frac{\sqrt{m}}{2(n-1)} + \frac{1}{2}, \quad (6)$$

then Φ can be used to reconstruct k -sparse signals via (P_1) . It is noted that (6) also greatly improves (5). The result in this paper can also be used to recover the sparse trigonometric polynomial with a single variable [27, 28].

In this paper, we also improve Katz' estimation for quadratic extension fields with an elementary and transparent approach. Using this improvement, we are able to construct a CS matrix which is a partial Fourier matrix, and whose columns are a union of orthonormal bases. This is a useful construction for sparse representation of signals in a union of orthonormal bases which has been a topic of some studies (see, for example [14, 15, 17, 18]). Moreover, this construction produces an approximately mutually unbiased bases which is of particular interest in quantum information theory. We also conduct some numerical experiments. The results show that the deterministic partial Fourier matrices has a better performance over the random partial Fourier matrices, provided that these two classes of matrices are of comparable sizes.

This paper is organized as follows. The section below provides some necessary concepts and results to be used in our discussion. The main results are given in section 3. The discussion of computational issues and numerical results are contained in the last section.

2 Background and Preparation

2.1 MIP and Sparse Recovery

Let Φ be an $m \times N$ matrix with normalized column vectors $\Phi_1, \Phi_2, \dots, \Phi_N$. Assume that each Φ_i is of unit (Euclidean) length. The mutual incoherence constant (MIC) is defined as

$$\mu = \max_{i \neq j} |\langle \Phi_i, \Phi_j \rangle|.$$

Even though in many situations a small μ is desired, the following well-known result of Welch [34] indicates that μ is bounded below

$$\mu \geq \sqrt{\frac{N-m}{(N-1)m}}.$$

In [14], Donoho and Huo gave a computationally verifiable condition on the parameter μ that ensures the sparse signal recovery: let Φ be a concatenation of two orthonormal matrices. Assume β is k -sparse. If

$$\mu < \frac{1}{2k-1}, \quad (7)$$

then the solution $\hat{\beta}$ for (P_1) is exactly β .

This result of [14] was extended to a general matrix Φ by Fuchs [16], and, Gribonval and Nielsen [18], both in the noiseless case. For the noisy case of the bounded error, [13, 5, 33] proved that ℓ_1 -minimization gives a stable approximation of the signal β under some conditions that are stronger than (7). The open problem of whether condition (7), namely $\mu < \frac{1}{2k-1}$, is sufficient for stable approximation of β in the noisy case was settled by Cai, Wang and Xu in [4]—actually the authors of [4] even proved that this condition is sharp, for both noisy and noiseless cases. Another remark is that if one considers only the noiseless case, the proof in [4] simplifies that of [14, 16, 18]

Although the sparse recovery condition (7) is rather strong, it is advantageous in checking whether a matrix meets the condition. Such a checking procedure requires $O(N^2)$ steps which is computationally feasible. The MIC μ has been explicitly used in the design of compressed sensing matrices by several works, e.g., [3, 12, 2, 24, 37].

2.2 RIP and Sparse Recovery

We say that Φ satisfies the Restricted Isometry Property (RIP) of order k and constant $\delta_k \in [0, 1)$ if

$$(1 - \delta_k)\|x\|_2^2 \leq \|\Phi x\|_2^2 \leq (1 + \delta_k)\|x\|_2^2 \quad (8)$$

holds for all k -sparse vector x (see [8]). In fact, (8) is equivalent to requiring that the Gramian matrices $\Phi_T^\top \Phi_T$ has all of its eigenvalues in $[1 - \delta_k, 1 + \delta_k]$ for all T with $|T| \leq k$, where Φ_T is the submatrix of Φ whose columns are those with indexes in T . It has been shown that, under various conditions on RIP constant δ_k , such as $\delta_k < \frac{1}{3}$ (see [6]), one can recover the k -sparse signal by solving (P_1) .

2.3 Katz' Characters Sums Estimation and Its Improvement

Let \mathbb{F}_{q^n} be a finite field of order q^n where q is a prime power. A multiplicative character χ is a homomorphism from the multiplicative group $\langle \mathbb{F}_{q^n}^*, \cdot \rangle$ to \mathbb{S}^1 where $\mathbb{S}^1 = \{z \in \mathbb{C} : \|z\| = 1\}$.

By a trivial character we mean the function that sends every element of $\mathbb{F}_{q^n}^*$ to 1.

Let g be a primitive root of \mathbb{F}_{q^n} (namely, g generates the multiplicative group $\mathbb{F}_{q^n}^*$). For each element $u \in \mathbb{F}_{q^n}^*$, we define the discrete logarithm of u with respect to the base g to be the non-negative integer m ($m < q^n - 1$) such that $g^m = u$, and we write $\log_g u = m$.

Let $N = q^n - 1$. Then a nontrivial multiplicative character of $\mathbb{F}_{q^n}^*$ is of the form

$$\chi_a(u) = e^{a \frac{2\pi i \log_g u}{N}}$$

where $1 \leq a \leq N - 1$.

For a nontrivial multiplicative character, a celebrated theorem of Katz [20] concerns the magnitude of summation of the character values over a special coset of \mathbb{F}_q (as an additive subgroup of \mathbb{F}_{q^n}). More precisely, Katz proved that for a nontrivial multiplicative character χ_a of $\mathbb{F}_{q^n}^*$, and for an element α of \mathbb{F}_{q^n} with $\mathbb{F}_{q^n} = \mathbb{F}_q(\alpha)$, there is an estimation

$$\left| \sum_{t \in \mathbb{F}_q} \chi_a(t - \alpha) \right| \leq (n - 1)\sqrt{q}. \quad (9)$$

The method that Katz used in [20] to obtain the estimate (9) was a geometric one. In [25], Li presented an arithmetic proof of (9) using the Riemann hypothesis for the projective line over finite fields. It is interesting to note, in the case of quadratic extension of the fields, we can get a more precise estimation of Katz sum with an elementary and transparent proof. This result will be a useful tool in our construction of special CS matrices. Our improvement of Katz estimation is stated as

Theorem 2.1 *Suppose that χ_a is a nontrivial multiplicative character of $\mathbb{F}_{q^2}^*$. For an element α of \mathbb{F}_{q^2} with $\mathbb{F}_{q^2} = \mathbb{F}_q(\alpha)$, we have*

$$\left| \sum_{t \in \mathbb{F}_q} \chi_a(t - \alpha) \right| = \sqrt{q}, \quad \text{if } (q - 1) \nmid a, \quad (10)$$

$$\sum_{t \in \mathbb{F}_q} \chi_a(t - \alpha) = -1, \quad \text{if } (q - 1) \mid a. \quad (11)$$

Proof. To this end, we first prove that

$$\sum_{t \in \mathbb{F}_q^*} \chi_a(t) = \begin{cases} 0, & q - 1 \nmid a \\ q - 1, & q - 1 \mid a \end{cases}. \quad (12)$$

Let g be a primitive root in \mathbb{F}_{q^2} . For $0 \leq m \leq q - 2$, the fact that $(g^{m(q+1)})^{(q-1)} = g^{m(q^2-1)} = 1$ implies

$$\{g^{m(q+1)} : 0 \leq m \leq q - 2\} = \mathbb{F}_q^*,$$

or, equivalently

$$(q+1) \mid \log_g t \iff t \in \mathbb{F}_q^*. \quad (13)$$

Therefore, (12) follows.

Now assume that $(q-1) \mid a$. We have

$$\begin{aligned} 0 &= \sum_{t \in \mathbb{F}_q^*} \chi_a(t) = \sum_{t_1, t_2 \in \mathbb{F}_q, (t_1, t_2) \neq (0,0)} \chi_a(t_1 + t_2 \alpha) \\ &= \sum_{t \in \mathbb{F}_q^*} \chi_a(t) + \sum_{t_2 \in \mathbb{F}_q^*} \sum_{t_1 \in \mathbb{F}_q} \chi_a(t_1 + t_2 \alpha) = \sum_{t \in \mathbb{F}_q^*} \chi_a(t) + \sum_{t_2 \in \mathbb{F}_q^*} \chi_a(t_2) \sum_{t_1 \in \mathbb{F}_q} \chi_a(t_2^{-1} t_1 + \alpha) \\ &= \sum_{t \in \mathbb{F}_q^*} \chi_a(t) + \sum_{t_2 \in \mathbb{F}_q^*} \chi_a(t_2) \sum_{t \in \mathbb{F}_q} \chi_a(t + \alpha) \\ &= \left(1 + \sum_{t \in \mathbb{F}_q} \chi_a(t + \alpha) \right) \sum_{t \in \mathbb{F}_q^*} \chi_a(t) = \left(1 + \sum_{t \in \mathbb{F}_q} \chi_a(t + \alpha) \right) (q-1), \end{aligned}$$

and hence $\sum_{t \in \mathbb{F}_q} \chi_a(t + \alpha) = -1$.

Next we consider the case $(q-1) \nmid a$ and we shall prove (10). It is not difficult to check that $\frac{t_1 - \alpha}{t_2 - \alpha} \notin \mathbb{F}_q^*$ for $t_1, t_2 \in \mathbb{F}_q$ and $t_1 \neq t_2$. In fact, if $\frac{t_1 - \alpha}{t_2 - \alpha} = s \in \mathbb{F}_q$, then $s \neq 1$. Therefore $\alpha = \frac{st_2 - t_1}{s - 1} \in \mathbb{F}_q$. This is absurd as $\alpha \notin \mathbb{F}_q$. Therefore by (13) we see that

$$(q+1) \nmid \log_g \frac{t_1 - \alpha}{t_2 - \alpha},$$

provided $t_1, t_2 \in \mathbb{F}_q$ and $t_1 \neq t_2$. Now

$$\begin{aligned} \left| \sum_{t \in \mathbb{F}_q} \chi_a(t - \alpha) \right|^2 &= \left| \sum_{t \in \mathbb{F}_q} e^{a \frac{2\pi i \log_g(t - \alpha)}{q^2 - 1}} \right|^2 = \left(\sum_{t_1 \in \mathbb{F}_q} e^{a \frac{2\pi i \log_g(t_1 - \alpha)}{q^2 - 1}} \right) \left(\sum_{t_2 \in \mathbb{F}_q} e^{a \frac{-2\pi i \log_g(t_2 - \alpha)}{q^2 - 1}} \right) \\ &= q + \sum_{t_1, t_2 \in \mathbb{F}_q, t_1 \neq t_2} e^{a \frac{2\pi i \log_g \frac{t_1 - \alpha}{t_2 - \alpha}}{q^2 - 1}} = q + \sum_{k=1, q+1 \nmid k}^{q^2-2} e^{a \frac{2\pi i k}{q^2 - 1}} \\ &= q + \sum_{k=1}^{q^2-2} e^{a \frac{2\pi i k}{q^2 - 1}} - \sum_{j=1}^{q-2} e^{a \frac{2\pi i j(q+1)}{q^2 - 1}} = q + \sum_{k=0}^{q^2-2} e^{a \frac{2\pi i k}{q^2 - 1}} - \sum_{j=0}^{q-2} e^{a \frac{2\pi i j(q+1)}{q^2 - 1}} \\ &= q, \end{aligned}$$

which yields

$$\left| \sum_{t \in \mathbb{F}_q} \chi_a(t - \alpha) \right| = \sqrt{q}.$$

■

3 The Main Results and Construction Procedure

In this section, we shall discuss construction of CS matrices by deterministically selecting set of rows from the $N \times N$ Fourier matrix.

In the following discussion, we let F_0, F_1, \dots, F_{N-1} be the rows of the $N \times N$ Fourier matrix $\mathcal{F}^{(N)}$, i.e.,

$$\mathcal{F}^{(N)} = \begin{pmatrix} F_0 \\ F_1 \\ \vdots \\ F_{N-1} \end{pmatrix}.$$

Suppose that M is a subset of $\{0, 1, \dots, N-1\}$, i.e., $M = \{m_0, m_1, \dots, m_r\} \subset \{0, 1, \dots, N-1\}$. Then we can define the partial Fourier matrix associated with M as

$$\mathcal{F}_M^{(N)} := \begin{pmatrix} F_{m_0} \\ F_{m_1} \\ \vdots \\ F_{m_r} \end{pmatrix}.$$

Let q be a prime power and let $\alpha \in \mathbb{F}_{q^n}$ be such that

$$\mathbb{F}_{q^n} = \mathbb{F}_q(\alpha).$$

Assume that g is a generator of the cyclic group $\mathbb{F}_{q^n}^*$, we then have

Theorem 3.1 *Let q be a prime power, and $n > 1$ be a positive integer. Let $N = q^n - 1$ and*

$$M = \{m = \log_g(t - \alpha) : t \in \mathbb{F}_q\}.$$

Then the $q \times N$ matrix $\Phi = \frac{1}{\sqrt{q}} \mathcal{F}_M^{(N)}$ has MIC

$$\mu \leq \frac{n-1}{\sqrt{q}}.$$

Proof. For $0 \leq j, k \leq N-1$ and $j \neq k$, the inner product of the j th and k th columns of the matrix Φ is

$$\begin{aligned} \langle \Phi_j, \Phi_k \rangle &= \frac{1}{q} \sum_{r=0}^{q-1} e^{\frac{2\pi j m_r i}{N}} e^{-\frac{2\pi k m_r i}{N}} = \frac{1}{q} \sum_{r=0}^{q-1} e^{(j-k) \frac{2\pi m_r i}{N}} \\ &= \frac{1}{q} \sum_{t \in \mathbb{F}_q} e^{(j-k) \frac{2\pi \log_g(t-\alpha) i}{N}} \end{aligned}$$

Since $j \neq k$, $\chi(u) = e^{(j-k)\frac{2\pi \log_g(u)i}{N}}$ defines a nontrivial multiplicative character of $\mathbb{F}_{q^n}^*$. By the Katz estimation (9) we have

$$\mu = \max_{j \neq k} |\langle \Phi_j, \Phi_k \rangle| = \frac{1}{q} \left| \sum_{t \in \mathbb{F}_q} \chi(t - x) \right| \leq \frac{n-1}{\sqrt{q}}.$$

■

We would like to point out that this result was also mentioned in [2]. However, Theorem 3.1 only produces a $q \times (q^n - 1)$ matrix and hence the matrix size in Theorem 3.1 is quite restrictive. Note that in DeVore's construction [12], the matrix size is $p^2 \times p^n$ where p is a prime. But Theorem 3.1 cannot always be used to produce matrices of size $\Theta(p^2 \times p^n)$, e.g., when n is an odd number.

We can have more flexibilities in choosing the size N of the Fourier matrix $\mathcal{F}^{(N)}$. This provides a larger set of parameters for constructing partial Fourier matrices for compressed sensing. More specifically, we have the following:

Theorem 3.2 *Let $q = p^a$ where p is a prime number. Suppose that b is a positive integer with $b \mid a$. Let $N = \frac{q^n - 1}{p^b - 1}$ where $n > 1$ is a positive integer and let*

$$M = \{m_k = \log_q(t_k - \alpha) \pmod{N} : t_k \in \mathbb{F}_q, k = 0, \dots, q-1\}.$$

Then the MIC for the $q \times N$ matrix $\Phi = \frac{1}{\sqrt{q}} \mathcal{F}_M^{(N)}$ satisfies

$$\mu \leq \frac{n-1}{\sqrt{q}}.$$

Proof. To this end, we first prove that $\#M = q$, i.e., $m_k \neq m_j$ if $k \neq j$. We assume that, for some $k \neq j$, $m_k = m_j$ holds. Then

$$\log_q(t_k - \alpha) \equiv \log_q(t_j - \alpha) \pmod{N}.$$

This implies that

$$\frac{t_k - \alpha}{t_j - \alpha} = g^{dN} \tag{14}$$

for some integer d . Obviously $g^{dN} \neq 1$ as $t_k \neq t_j$. Noting that $(g^{dN})^{p^b-1} = (g^{q^n-1})^d = 1$, we see that

$$g^{dN} \in \mathbb{F}_{p^b} \subset \mathbb{F}_q.$$

Solving (14) for α , we get

$$\alpha = \frac{g^{dN} t_j - t_k}{g^{dN} - 1} \in \mathbb{F}_q,$$

which implies that $\mathbb{F}_q(\alpha) = \mathbb{F}_q$. This is impossible as $n > 1$.

Now we consider the $q \times N$ matrix Φ . The inner product of the j th and k th columns of the matrix Φ is

$$\begin{aligned}\langle \Phi_j, \Phi_k \rangle &= \frac{1}{q} \sum_{m \in M} e^{\frac{2\pi jmi}{N}} e^{-\frac{2\pi kmi}{N}} = \frac{1}{q} \sum_{m \in M} e^{(j-k)(p^b-1)\frac{2\pi mi}{N}} \\ &= \frac{1}{q} \sum_{t \in \mathbb{F}_q} e^{(j-k)(p^b-1)\frac{2\pi \log_q(t-\alpha)i}{N}},\end{aligned}$$

where $\tilde{N} = N(p^b - 1) = q^n - 1$. Since for $0 \leq j, k < N$ and $j \neq k$,

$$\chi(u) = e^{(j-k)(p^b-1)\frac{2\pi \log_q(u)i}{N}}$$

defines a nontrivial multiplicative character of $\mathbb{F}_{q^n}^*$. Again, by the Katz estimation (9), we arrive at

$$\mu = \max_{j \neq k} |\langle \Phi_j, \Phi_k \rangle| = \frac{1}{q} \left| \sum_{t \in \mathbb{F}_q} \chi(t - \alpha) \right| \leq \frac{n-1}{\sqrt{q}}.$$

■

Remark 3.1 1. According to our discussion in the previous section, a k -sparse signal β in model (1) can be reconstructed via the ℓ_1 -minimization (P_1) as long as $\mu < \frac{1}{2k-1}$. The matrix Φ in Theorem 3.1 and Theorem 3.2, can be a CS matrix for recovering k sparse signals if

$$k < \frac{\sqrt{q}}{2(n-1)} + \frac{1}{2}.$$

This bound is the same as the case of binary CS matrices obtained by DeVore [12]. Also, this bound improves the one for subsampling Fourier matrices in [19] greatly.

2. In terms of matrix dimension, our theorems provide more flexibilities. In [12], the (binary) CS matrices can be of size $p^2 \times p^r$ for any $r \geq 2$. We can get similar dimensions for partial Fourier CS matrices too. Taking $q = p^2$ in theorem 3.1, we can get a $p^2 \times \Theta(p^{2n})$ matrix; letting $a = 2$ and $b = 1$ in theorem 3.2, we have a $p^2 \times \Theta(p^{2n-1})$ matrix.
3. For the number $N = \frac{q^n-1}{p^b-1} = \frac{p^{an}-1}{p^b-1}$, as pointed out in [3], the coherence of a $q \times N$ random matrix with i.i.d. Gaussian entries is about $2\sqrt{\frac{\log N}{q}} \approx 2\sqrt{\frac{(an-b)\log p}{q}}$. Hence, when $\log p > \frac{(n-1)^2}{4(an-b)}$, MIC of the deterministic matrix Φ given in Theorem 3.2 is smaller than that of random matrices.

Finally in this section, we restrict ourselves to the case of quadratic extension. We shall construct a partial Fourier matrix whose columns form a union of orthonormal bases, by using our improvement of Katz' estimation. The construction of such matrix is also raised in quantum information theory

Let q be a prime power and let α be such that $\mathbb{F}_{q^2} = \mathbb{F}_q(\alpha)$. As before, we let $N = q^2 - 1$ and denote

$$M = \{m = \log_q(t - \alpha) : t \in \mathbb{F}_q\}. \quad (15)$$

It is easy to see that $0 \notin M$. We are able to state

Theorem 3.3 *Let*

$$\Phi = \frac{1}{\sqrt{q+1}} \mathcal{F}_{M \cup \{0\}}^{(N)} \in \mathbb{C}^{(q+1) \times (q^2-1)}, \quad (16)$$

where M is defined in (15). For each $j = 0, \dots, q-2$, set

$$T_j = \{j + k \cdot (q-1) : 0 \leq k \leq q\}.$$

Then we have

1. For any $0 \leq j \leq q-2$, Φ_{T_j} is an orthogonal matrix.
2. For $k_1 \in T_{j_1}$ and $k_2 \in T_{j_2}$ with $j_1 \neq j_2$, we have

$$\frac{\sqrt{q}-1}{q+1} \leq |\langle \Phi_{k_1}, \Phi_{k_2} \rangle| \leq \frac{\sqrt{q}+1}{q+1}$$

Proof. For any $k_1, k_2 \in T_j$ with $k_1 \neq k_2$. Since $(q-1) \mid k_1 - k_2$, using (11) of theorem 2.1 we get

$$\begin{aligned} \langle \Phi_{k_1}, \Phi_{k_2} \rangle &= \frac{1}{q+1} \left(1 + \sum_{r=0}^{q-1} e^{\frac{2\pi k_1 m_r i}{N}} e^{-\frac{2\pi k_2 m_r i}{N}} \right) = \frac{1}{q+1} \left(1 + \sum_{r=0}^{q-1} e^{(k_1 - k_2) \frac{2\pi m_r i}{N}} \right) \\ &= \frac{1}{q+1} (1 + (-1)) = 0. \end{aligned}$$

This shows Φ_{T_j} is an orthonormal matrix for any $0 \leq j \leq q-2$.

We next consider the inner product of Φ_{k_1}, Φ_{k_2} with k_1 and k_2 belonging to different sets $T_j, j = 0, \dots, q-2$. We have

$$\begin{aligned} |\langle \Phi_{k_1}, \Phi_{k_2} \rangle| &= \frac{1}{q+1} \left| 1 + \sum_{r=0}^{q-1} e^{\frac{2\pi k_1 m_r i}{N}} e^{-\frac{2\pi k_2 m_r i}{N}} \right| \\ &= \frac{1}{q+1} \left| 1 + \sum_{r=0}^{q-1} e^{\frac{2\pi (k_1 - k_2) m_r i}{N}} \right| \end{aligned}$$

Since $(q-1) \nmid k_1 - k_2$, using (10) of theorem 2.1 we get

$$\frac{\sqrt{q}-1}{q+1} \leq |\langle \Phi_{k_1}, \Phi_{k_2} \rangle| \leq \frac{\sqrt{q}+1}{q+1}.$$

■

Remark 3.2 *The concept of approximately mutually unbiased bases (AMUBs) (see [21, 22, 23]) arises from quantum information theory. An $m \times N$ matrix Φ (with columns of unit length and with $m \mid N$) is a AMUBs if (i) the set of columns of Φ can be partitioned into $\frac{N}{m}$ sets of m vectors, and each set is an orthonormal basis for \mathbb{C}^m ; (ii) $|\langle \Phi_{k_1}, \Phi_{k_2} \rangle| = \sqrt{\frac{1}{m}} + o(1)$ holds if Φ_{k_1} and Φ_{k_2} are taken from different bases. By Taylor expansion, we have*

$$\frac{\sqrt{q}+1}{q+1} = \sqrt{\frac{1}{q+1}} + O\left(\frac{1}{q+1}\right), \quad \frac{\sqrt{q}-1}{q+1} = \sqrt{\frac{1}{q+1}} - O\left(\frac{1}{q+1}\right).$$

So Theorem 3.3 says that we have actually constructed an AMUBs.

4 Computational Issues and Numerical Example

In this section we will first discuss the issues of finding a primitive root g in \mathbb{F}_{q^n} and of computing discrete logarithm with respect to g . We then explain some results from our numerical experiments.

4.1 Computational Issues

Finding a primitive root of a finite field is an interesting problem. In [32], Shparlinski gave a deterministic algorithm which returns a primitive root g of \mathbb{F}_{q^n} in time $O(q^{\frac{n}{4}+\epsilon})$. It is noted that there is no polynomial time algorithm available for this problem even under the General Riemann Hypothesis unless q is prime and $n \leq 2$ (see Shoup [31]). The idea of Shparlinski's algorithm is to construct a subset $M \subset \mathbb{F}_{q^n}^*$ such that

1. M contains a primitive root;
2. $|M| = O(q^{\frac{n}{4}})$.

To identify a primitive root in M one just uses the following fact: an element $g \in \mathbb{F}_{q^n}^*$ is a primitive root if and only if $g^{\frac{q^n-1}{\ell}} \neq 1$ for any prime factor ℓ of $q^n - 1$.

Computing a discrete logarithm over a finite field is also of great importance because the hardness of this problem is a base of several well-known cryptosystems. The index calculus methods provide efficient ways of solving discrete logarithm problem over finite fields (see [26]). However, since in our setting, the size of the finite field involved is far smaller than that in the cryptographical setting, we can use the following algorithm of Shanks [30] to compute the discrete logarithm in $\mathbb{F}_{q^n}^*$. Again, denote $N = q^n - 1$ and let $K = \lceil \sqrt{N} \rceil$.

Computing $\log_g u$ for $u \in \mathbb{F}_{q^n}^*$

- 1 For all $0 \leq j < K$, compute g^j and store (j, g^j) in list L .
- 2 Compute g^{-K}
- 3 For $i = 0$ to $K - 1$
 - Compute $u(g^{-K})^i$
 - if $u(g^{-K})^i$ equals some g^j in the list L , return $\log_g u = iK + j$.

The computational costs are $O(\sqrt{N})$ both in time and space.

4.2 Numerical Examples

The purpose of the experiments is to provide some performance comparisons of the random sampling and the deterministic sampling of Fourier matrices. We use the method introduced in this paper to produce a deterministic partial Fourier matrix. To compare with the deterministic partial Fourier matrix, we generate the random partial Fourier matrix by choosing the same number of rows (as that in the deterministic case) from the Fourier matrix in a uniformly random manner. We reconstruct the Fourier coefficients by OMP (see [27, 36]) and BP, respectively. For BP, we use the optimization tools of CVX [7]. We repeat the experiment 100 times for each sparsity $k \in \{1, \dots, 20\}$ and calculate the success rate in Example 1 and Example 2. In Example 3, we compare the maximum and minimum eigenvalue statistics of Gram matrices $\Phi_T^T \Phi_T$ of varying sparsity $k = |T|$ for a deterministic sampling matrix and a random sampling matrix. It is interesting to note that all the numerical results show that the deterministic partial Fourier matrix has a better performance over the random partial Fourier matrix, regardless whether the sparsity is within the theoretical range (i.e. $k < \frac{1}{2} \left(\frac{1}{\mu} + 1 \right)$) or not.

Example 1 Take $q = 29$ and $N = q^2 - 1 = 840$. Consider the field $\mathbb{F}_{29^2} = \mathbb{F}_{29}[x]/(x^2 + 2)$. Denote $\bar{x} = x + (x^2 + 2)$ and choose $\alpha = \bar{x}, g = \bar{x} + 2$. We have

$$\begin{aligned} M_1 &= \{\log_g(t - \alpha) : t \in \mathbb{F}_{29}\} \\ &= \{465, 1, 494, 649, 47, 507, 758, 610, 835, 244, 67, 204, 588, 519, \\ &\quad 332, 808, 351, 672, 456, 683, 776, 275, 470, 562, 3, 103, 761, 466, 449\}, \end{aligned}$$

By Theorem 3.1, we can build a 29×840 partial Fourier matrix $\Phi = \frac{1}{\sqrt{q}} \mathcal{F}_{M_1}^{(N)}$. To compare, we generate the random partial Fourier matrix selecting 29 rows from the 840×840 Fourier matrix uniformly randomly.

Figure 1 depicts the recovery results with using the recovery algorithm OMP and BP, respectively.

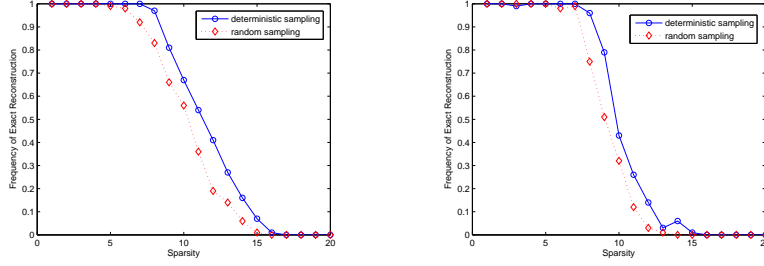


Figure 1: Numerical experiments for the comparison of the random and the deterministic 29×840 partial Fourier matrix. The left graph corresponds to the success rates recovering by OMP, whereas the right one depicts the success rate of BP.

Example 2 We take $q = p = 19$ and $N = \frac{p^3-1}{p-1} = p^2 + p + 1 = 381$. Then we have

$$\begin{aligned} M_2 &= \{\log_g(t - \alpha) \bmod N : t \in \mathbb{F}_{19}\} \\ &= \{192, 208, 162, 165, 160, 39, 154, 141, 245, 356, 304, 311, 223, 40, 321, 68, 118, 174, 249\}, \end{aligned}$$

where $\mathbb{F}_{19^3} = \mathbb{F}_{19}[x]/(x^3 + x + 1)$, $\alpha = \bar{x}$, $g = \bar{x}^2 + 2\bar{x}$. By Theorem 3.2, we can build a 19×381 sub-Fourier matrix $\Phi = \frac{1}{\sqrt{q}}\mathcal{F}_{M_2}^{(N)}$. Similar to Example 1, we also compare the deterministic sampling and the random sampling and show the result in Figure 2.

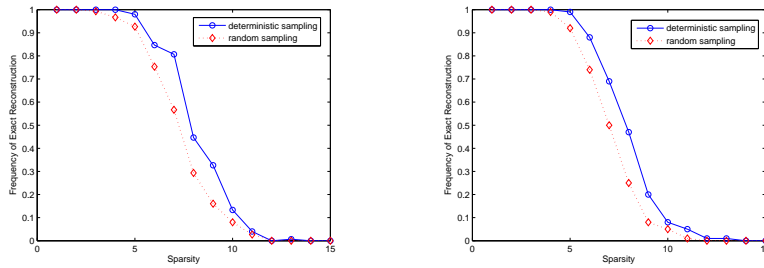


Figure 2: Numerical experiments for the comparison of the random and the deterministic 19×381 partial Fourier matrix. The left graph corresponds to the success rates recovered by OMP, whereas the right one depicts the success rate of BP.

Example 3 The aim of the numerical experiment is to compare the maximum and minimum eigenvalue statistics of Gram matrices $\Phi_T^\top \Phi_T$ of varying sparsity $M = |T|$ for deterministic sampling matrix and random sampling matrix. In fact, RIP of order k with constant δ_k is equivalent to requiring that the Grammian matrices $\Phi_T^\top \Phi_T$ has all of its eigenvalues in $[1 - \delta_k, 1 + \delta_k]$ for all T with $|T| \leq k$. For every value M , sets T are drawn

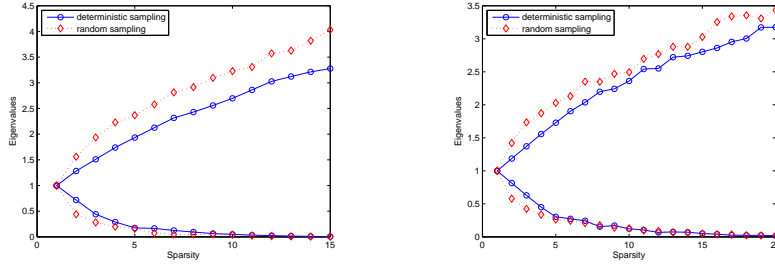


Figure 3: Eigenvalue statistics of Gram matrices $\Phi_T^\top \Phi_T$ for the deterministic sampling matrixes and random sampling matrixes. The left graph shows the result of 19×381 partial Fourier matrix in Example 2, whereas the right one depicts the result of 29×840 partial Fourier matrix in Example 1.

uniformly random over all sets and the statistics are accumulated from 50,000 samples. Figure 3 shows the maximum and minimum eigenvalues of $\Phi_T^\top \Phi_T$ for $\#T \in \{1, \dots, 20\}$.

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